

The structure of stable minimal hypersurfaces in \mathbb{R}^{n+1}

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Abstract

We provide a new topological obstruction for complete stable minimal hypersurfaces in \mathbb{R}^{n+1} . For $n \geq 3$, we prove that a complete orientable stable minimal hypersurface in \mathbb{R}^{n+1} cannot have more than one end by showing the existence of a bounded harmonic function based on the Sobolev inequality for minimal submanifolds [MS] and by applying the Liouville theorem for harmonic functions due to Schoen-Yau [SY].

1 Introduction

This paper is concerned with the structure of complete stable minimal hypersurfaces in $(n+1)$ -dimensional Euclidean space \mathbb{R}^{n+1} . A complete oriented minimal submanifold M in \mathbb{R}^{n+1} is called stable if the second variation of the volume is non-negative on any compact subset of M . The fundamental result along these lines is the Bernstein Theorem [B] which says that a complete area-minimizing graph in \mathbb{R}^3 is a plane. Much work has been devoted to trying to generalize it in the last thirty years. From the works of Fleming [F], De Giorgi [DG], Almgren [A] and J. Simons [SJ], one knows that Bernstein Theorem is valid for complete area-minimizing graphs in \mathbb{R}^{n+1} for $n \leq 7$. Counterexamples to the theorem for $n \geq 8$ were found by Bombieri-De Giorgi-Giusti [BDG] and later by Lawson [L]. Since then, there have been attempts to extend the above Bernstein Theorem by assuming that the minimal hypersurface be stable. The best result in this direction is due to Fischer-Colbrie-Schoen [FS] and do Carmo-Peng [DP], who proved that if

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M^2 is a complete, oriented and immersed stable minimal surface in \mathbb{R}^3 , then M is a plane.

However, not much is known for the geometric structure of stable minimal hypersurfaces in \mathbb{R}^{n+1} when $n \geq 3$. The only known topological obstruction for stable minimal submanifolds we are aware of was a result of Schoen-Yau [SY], which states that if M^n is a complete stable hypersurface in a manifold of non-negative curvature and D is a compact domain in M with smooth simply connected boundary, then there is no non-trivial homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold with nonpositive curvature. We remark that Schoen-Yau [SY] proved the same result when M is a complete manifold with non-negative Ricci curvature. In this paper we provide a new topological obstruction for complete stable minimal hypersurfaces in \mathbb{R}^{n+1} . Our main result can be stated as follows:

Theorem 1. *For any $n \geq 3$, if M^n is a complete non-compact oriented stable minimal hypersurface in \mathbb{R}^{n+1} , then M has only one end.*

To our knowledge, this result is new even for area-minimizing hypersurfaces M^n in \mathbb{R}^{n+1} for $n \geq 8$. Note that the stable condition in Theorem 1 cannot be dropped, since a catenoid, which is unstable, clearly has two ends. It is also clear that our result differs from Schoen-Yau's theorem. For example, our result says that any manifold of topological type $N^{n-1} \times \mathbb{R}$ with N^{n-1} compact cannot be a stable minimal hypersurface in \mathbb{R}^{n+1} . In fact, Theorem 1 can be compared with the similar result of Gromoll-Meyer [GM] for complete manifolds with positive Ricci curvature.

The proof of our main theorem relies on the Sobolev inequality for minimal submanifolds due to Michael and Simon [MS] and the Liouville theorem for harmonic maps due to Schoen and Yau [SY]. One crucial step in the proof is to show the existence of a non-trivial bounded harmonic function with finite Dirichlet energy in case the minimal hypersurface has more than one end. This is done by using the Sobolev inequality together with a choice of cut-off functions based on the fact that the minimal submanifold has more than one end. We remark that our cut-off function actually has noncompact support. This non-standard choice of cut-off functions allows us to avoid assumptions such as volume doubling properties and volume growth conditions of the ends.

Finally, we would like to point out that the method we used to in the

proof of Lemma 2 in next section yields the following result which is of independent interest:

Theorem 2. *Let M^n be a complete noncompact Riemannian manifold with at least two ends of infinite volume. Suppose that either*

- (1) the Sobolev inequality holds on M , or*
- (2) the first eigenvalue $\lambda_1(M)$ of M is positive.*

Then there exists on M a non-constant bounded harmonic function with finite Dirichlet energy.

The problem of the existence of harmonic functions on a complete manifold has a long history. For some of the recent progress on this problem, we refer the readers to the important works of Li-Tam ([LT1] and [LT2]), Colding-Minicozzi ([CM1], [CM2], [CM3] and [CM4]) and a very recent survey article of Peter Li [Li].

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2 The Proof

In this section we prove Theorem 1 stated in the introduction. First let us fix some notations and recall the definition of a stable minimal submanifold.

Let $\{U_i\}_{i=1}^\infty$ be a family of relatively compact open sets which exhaust the manifold M , i.e.,

$$\begin{aligned} U_i &\subset U_{i+1}, \\ \cup_{i=1}^\infty U_i &= M. \end{aligned}$$

An end of M is an inverse system $E = \{E^{(i)}\}_{i=1}^\infty$ such that

$$E^{(i+1)} \subset E^{(i)}$$

and $E^{(i)}$ is a connected component of $M \setminus \overline{U_i}$.

Remark 2. If M has only finitely many ends, then there is some $i_0 > 0$ such that all inverse systems $(E^{(i)})$ stabilize for $i \geq i_0$, i.e., $E^{(i)} = E^{(i_0)}$.

Remark 3. An end can also be defined as equivalent classes of cofinal curves, where two curves $c_1, c_2: [0, \infty) \rightarrow M$ are cofinal iff for every compact set $K \subset M$ there is some $t > 0$ such that $c_1(t_1)$ and $c_2(t_2)$ lie in the same connected component of $M \setminus K$ for all $t_1, t_2 > t$.

Let M^n be a completed oriented submanifold minimally immersed in \mathbb{R}^{n+1} . We say that M is stable if the second variation of the volume is non-negative on any compact subset of M . More precisely, let e_1, e_2, \dots, e_{n+1} be a positively oriented orthonormal frame on M with e_1, \dots, e_n tangential, and e_{n+1} the globally defined unit normal vector to M . We can define the second fundamental form $\{h_{ij}\}$ of M by

$$h_{ij} = \langle \nabla_{e_i} e_{n+1}, e_j \rangle$$

for $i, j = 1, 2, \dots, n$, where ∇ is the Riemannian connection of \mathbb{R}^{n+1} . Then M is minimal iff the mean curvature

$$H = \sum_{i=1}^n h_{ii} = 0.$$

The stability of M is given by the following inequality (see [SJ] or [SY]):

$$\int_M |\nabla \phi|^2 \geq \int_M \sum_{i,j=1}^n h_{ij}^2 \phi^2$$

where ϕ is any function with compact support on M .

Now we are ready to present the proof of Theorem 1.

Lemma 1. *If M^n is a complete orientable minimal hypersurface in \mathbb{R}^{n+1} , then every end of M^n has infinite volume.*

Proof: In fact we will show that for any compact set $K \subset M$, every noncompact component of $M \setminus K$ has infinite volume. Let E be a component of $M \setminus K$. We will adopt an argument of Yau ([Y]) to the end E .

Take an arbitrary point $p \in M$, without loss of generality, we may assume $p = 0$. In the following we let $d(\cdot, \cdot)$ be the distance function of \mathbb{R}^{n+1} , and $r(\cdot, \cdot)$ the distance function of M with respect to the induced metric. We will

write $d(x), r(x)$ if the base point is 0. Obviously $d \leq r$ for any two points in M . Let γ be a minimal geodesic from 0, then,

$$\begin{aligned} \frac{\partial d}{\partial r} &= \lim_{t \rightarrow 0} \frac{d(\gamma(s+t)) - d(\gamma(s))}{t} \\ &\leq \lim_{t \rightarrow 0} \frac{d(\gamma(s+t), \gamma(s))}{t} \quad (\text{by the triangle inequality}) \\ &\leq 1. \quad (\text{since } d \leq r) \end{aligned} \tag{1}$$

By a direct computation and using the fact that M is minimal, one can show that

$$\Delta_M d^2(x) = 2n.$$

Let $B(s)$ be the geodesic ball of M , of radius s centered at 0. Integrating the above equation over $B(s)$ and using (1), we obtain

$$2n \operatorname{vol}(B(s)) \leq 2s \operatorname{vol}(\partial B(s)).$$

Note that in any manifold,

$$\operatorname{vol}(\partial B(s)) = \frac{\partial}{\partial r} \Big|_{r=s} \operatorname{vol}(B(s)).$$

We thus obtain

$$s \frac{\partial}{\partial r} \Big|_{r=s} \operatorname{vol}(B(r)) - n \operatorname{vol}(B(s)) \geq 0,$$

which implies $s^{-n} \operatorname{vol}(B(s))$ is nondecreasing. Therefore

$$\frac{\operatorname{vol}(B(s))}{s^n} \geq \lim_{s \rightarrow 0} \frac{\operatorname{vol}(B(s))}{s^n} = \omega(n)$$

where $\omega(n)$ is the volume of unit ball in \mathbb{R}^n .

Now if E has finite volume, choose R big enough such that

$$\omega(n)R^n > \operatorname{vol}(E).$$

Let p be a point in E such that $r(p, \partial E) \geq R$, then

$$\operatorname{vol}(E) \geq \operatorname{vol}(B(R)) \geq \omega(n)R^n > \operatorname{vol}(E),$$

a contradiction.

q.e.d

Lemma 2. *Let M^n be a complete orientable minimal hypersurface in \mathbb{R}^{n+1} with at least two ends. Then there exists on M a non-constant bounded harmonic function with finite energy.*

We remark that the statement of Lemma 2. still holds when M^n is a complete submanifold in \mathbb{R}^{n+m} for $n \geq 3, m \geq 1$ provided one of the following conditions holds: (1) $\int_M |\vec{H}|^n < 1/c_n^{\frac{n}{2}}$; or (2) $\int_M |\vec{H}|^{\frac{4n}{n+2}} < \infty$. Here \vec{H} is the mean curvature vector of M and c_n is a (Sobolev) constant which depends only on the dimension n .

Proof of Lemma 2: In the following, we will use an exhaustion of M by compact submanifolds with boundary. The obvious choice is the distance balls which are not smooth in general. It is standard to smooth the distance function to a point p by, say, local averaging, to get a smooth function f . f still has compact sublevel sets. By Sard's Theorem, we can choose a sequence of regular values $\{R_i\}$ of f , such that $\{f^{-1}(0, R_i)\}$ gives an exhaustion of M by compact smooth submanifolds. In the following, we denote $D_i = f^{-1}(0, R_i)$. We shall also use the notations for the ends of M introduced at the beginning of this section. For $i \geq i_0$ and i_0 sufficiently large, let

$$M \setminus D_i = \cup_{j=1}^s E_j^{(i)}$$

be the disjoint union of connected components, with $s \geq 2$.

By Lemma 1 and the assumptions in Lemma 2, M has at least two components with infinite volume. Let $E_1^{(i_0)}, E_2^{(i_0)}$ be two such components. On each compact domain D_i , we can minimize the energy functional $\int_{D_i} |\nabla u|^2 dx$ among all functions u such that $u|_{\partial E_1^{(i)}} = 1$ and $u|_{\partial E_j^{(i)}} = 0$ for all $j \geq 2$, where ∇ and dx are gradient and volume element of M , respectively. We denote the minimizer by u_i . Then u_i is the unique solution of the following Dirichlet problem on D_i

$$\begin{cases} \Delta u(x) &= 0 \\ u|_{\partial E_1^{(i)}} &= 1 \\ u|_{\partial E_j^{(i)}} &= 0. \quad (j \neq 1) \end{cases} \quad (2)$$

By maximum principle, we have $0 \leq u_i \leq 1$ on D_i . Moreover, it is easy to see that $\int_{D_i} |\nabla u_i|^2 dx \leq \int_{D_j} |\nabla u_j|^2 dx$ for $i > j$ and hence there is a universal

constant $C_1 > 0$ such that

$$\int_{D_i} |\nabla u_i|^2 dx < C_1. \quad (3)$$

Therefore by passing to a subsequence, still denoted by u_i , we can find a harmonic function u on M such that

$$\lim_{i \rightarrow \infty} u_i(x) = u(x), \quad x \in M$$

and

$$\int_M |\nabla u|^2 dx < C_1.$$

It is clear from the construction that $0 \leq u(x) \leq 1$ on M .

In the following we prove that the limiting harmonic function u is not a constant function. We will prove this by contradiction.

By using the Sobolev inequality for minimal hyperfaces in \mathbb{R}^{n+1} (see [MS]), we have, for any smooth function ϕ which vanishes on ∂D_i ,

$$\left(\int_{D_i} \phi^p(x) dx \right)^{\frac{2}{p}} \leq c_n \int_{D_i} |\nabla \phi|^2 dx \quad (4)$$

where $p = 2n/n - 2$ (this is where we need to assume $n \geq 3$) and c_n is the Sobolev constant which only depends on the dimesion n .

Note that from the construction of u_i , the function $u_i(1 - u_i)$ vanishes on ∂D_i . Setting $\phi = u_i(1 - u_i)$ in (4), we obtain

$$\begin{aligned} \left(\int_{D_i} (u_i(1 - u_i))^p dx \right)^{\frac{2}{p}} &\leq c_n \int_{D_i} (\nabla u_i - 2u_i \nabla u_i)^2 dx \\ &\leq c_n \int_{D_i} (|\nabla u_i| + 2u_i |\nabla u_i|)^2 dx \\ &\leq 9c_n \int_{D_i} |\nabla u_i|^2 dx \leq 9c_n C_1. \end{aligned} \quad (5)$$

Since $\text{vol}(D_i) \rightarrow \infty$, by letting $i \rightarrow \infty$ in (5), it follows that if u is a constant function, then $u \equiv 0$ or $u \equiv 1$.

Thus we only need to show that u cannot be identically 0 or 1. If $u \equiv 0$, we may replace u_i and u by $\hat{u}_i = 1 - u_i$ and $\hat{u} = 1 - u$, respectively. Then $\hat{u} \equiv 1$ and furthermore, \hat{u}_i satisfies (3), and (5). Thus we may assume that $u \equiv 1$. (This is where we use the condition that the manifold has at least two ends with infinite volume.)

We choose a smooth function ψ such that

$$\psi = \begin{cases} 1 & \text{in } E_2^{(i_0)} \\ 0 & \text{in } E_j^{(i_0)} \quad j \neq 2 \end{cases}$$

and

$$|\nabla \psi| \leq C_2, \quad 0 \leq \psi \leq 1$$

for some constant $C_2 > 0$ which is independent of i and u_i . Note that $|\nabla \psi|$ vanishes outside a compact set.

Since $u_i|_{E_1^{(i)}} = 1$ and $u_i|_{E_j^{(i)}} = 0$ for $j \geq 2$, the function $\phi = u_i\psi$ vanishes on ∂D_i . The Sobolev inequality (5) implies that

$$\begin{aligned} \left(\int_{D_i} (u_i\psi)^p dx \right)^{\frac{2}{p}} &\leq c_n \int_{D_i} |\nabla(u_i\psi)|^2 dx \\ &\leq c_n \int_{D_i} |\psi \nabla u_i + u_i |\nabla \psi||^2 dx \\ &\leq 2c_n \left(\int_{D_i} \psi^2 |\nabla u_i|^2 + \int_{D_i} |\nabla \psi|^2 \right) \leq C_3 \end{aligned} \quad (6)$$

where C_3 is a constant independent of i and u_i .

Therefore, we have

$$\int_{E_2^{(i_0)} \cap D_i} (u_i)^p \leq C_3$$

for all $i \geq i_0$. Letting $i \rightarrow \infty$, we get

$$\text{vol}(E_2^{(i_0)}) = \int_{E_2^{(i_0)}} u^p \leq C_3. \quad (7)$$

This contradicts our assumption that $\text{vol}(E_2^{(i_0)})$ is infinite. Therefore the limiting harmonic function u is not a constant function. q.e.d.

Lemma 3. (Schoen-Yau [SY]) *Let M be a complete noncompact stable minimal hypersurface in a manifold of non-negative curvature. If u is a harmonic function on M with bounded energy, then u is constant.*

This is a special case of the Liouville Theorem for harmonic map that Schoen and Yau originally proved.

Now suppose $M^n (n \geq 3)$ is a complete, oriented stable minimal hypersurface in \mathbb{R}^{n+1} with finitely many ends. From Lemma 1 we know that each end of M has infinite volume. If M has more than one end then Lemma 2 implies that M supports a non-constant harmonic function with finite energy. This is a contradiction to Lemma 3. Hence M has only one end and the proof of Theorem 1 is completed.

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